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# Random walks on unimodular $p$-adic groups

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**Abstract**

In a recent paper Pittet and Saloff-Coste established the lower bound $p_{2n}(e,e) \geqslant c \exp(-Cn^{1/3})$, $n = 1, 2, \ldots$ for the large times asymptotic behaviours of the probabilities $p_{2n}(e,e)$ of return to the origin at even times $2n$, for random walks associated with finite symmetric generating sets of solvable groups of finite Prüfer rank and asked if a similar lower bound is available in the case of the semi-direct product $\mathrm{Sol}(\mathbb{Q}_p) = \mathbb{Q}_p^2 \ltimes \mathbb{Q}_p^*$. In this paper, we give an answer to this problem.


*MSC:* 22E35; 60B15; 60G50

*Keywords:* Random walks; $p$-adic groups

## 1. Introduction

Let $G$ be a locally compact unimodular amenable group. Let $\mathrm{d}g$ denote the Haar measure on $G$ and let $\mathrm{d}\mu(g) = \varphi(g)\,\mathrm{d}g \in \mathbf{P}(G)$ be a probability measure on $G$, where $\varphi(g) \in L^\infty(G)$ is assumed to have a compact support or a fast decay at infinity. Let us assume that $\mu$ is symmetric (i.e. the involution $g \to g^{-1}$ stabilizes $\mu$) and consider the random walk on $G$ induced by $\mu$, i.e. the $G$-valued process that evolves as follows: If $X_n = g$ is the position at time $n$ then $X_{n+1} = gh$, where $h$ is chosen according to $\mu$. Let us denote by $\mathrm{d}\mu^{*n}(g) = \varphi_n(g)\,\mathrm{d}g$ the $n$th convolution power of $\mu$. One of the most

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basic quantity of interest in this context is $\varphi_n(e)$ (where $e$ is the identity of $G$). The problem is to decide how fast $\varphi_n(e) \to 0$ as $n \to \infty$.

If we restrict ourselves to real Lie groups then the answer lies in the behaviour of the volume growth of $G$. Let us recall that if $G$ is a locally compact group that is generated by some symmetric compact neighbourhood $\Omega \subset G$ of the identity $e$ in $G$, the volume growth function is (cf. [6])

$$\gamma(t) = \mathrm{Vol}(B_t(e)), \quad t = 1, 2, \ldots,$$

where the volume is taken with respect to the Haar measure $\mathrm{d}g$ and where $B_t(e)$ is the ball of radius $t$ centred on $e$ defined by

$$B_t(e) = \Omega \ldots \Omega, \quad t \text{ times.}$$

For $x \in G$ the distance from $e$ is defined by $|x| = \inf\{t, x \in B_t(e)\}$ and a left invariant distance can be defined on $G$ by $\mathrm{d}(x, y) = |y^{-1}x|$, $x, y \in G$. In the Lie groups case we can also use a left invariant Riemannian distance induced on $G$ by some fixed scalar product on $\mathrm{Lie}(G)$ to define the balls $B_t(e) = B_t^{\mathrm{Rim}}(e)$ and take $\gamma^{\mathrm{Rim}}(t) = \mathrm{Vol}(B_t^{\mathrm{Rim}}(e))$. It is easy to see that this new growth function and the previous one satisfy the obvious equivalence $\gamma(t) \approx \gamma^{\mathrm{Rim}}(t)$, i.e.

$$\gamma^{\mathrm{Rim}}(t) \leqslant C\gamma(Ct) + C \leqslant C\gamma^{\mathrm{Rim}}(Ct) + C, \quad t \geqslant 1.$$

For Lie groups we have the following dichotomy (cf. [6]): either

$$\gamma(t) \approx t^D$$

where $D = D(G) = 1, 2, \ldots,$ or

$$\gamma(t) \approx \mathrm{e}^t.$$

In the first case, we say that $G$ is of polynomial growth and in the second case we say that $G$ is of exponential growth and the answer to our problem in the case of real Lie groups is the following:

$$\varphi_n(e) \approx n^{-D/2} \iff \gamma(t) \approx t^D,$$

$$\varphi_n(e) \approx \mathrm{e}^{-n^{1/3}} \iff \gamma(t) \approx \mathrm{e}^t$$

(cf. [1,5,16]).

The discrete case is more complicated (cf. [1,5,7–10,13–15]). First, there is no dichotomy in the volume growth. On the other hand, if we suppose that the group $G$ is of exponential growth then one can claim only the upper bound

$$\varphi_n(e) \leqslant C \exp(-cn^{1/3}), \quad n \geqslant 1.$$

In general, the matching lower bound fails. Pittet and Saloff-Coste showed (cf. [8,9]) that there are soluble groups with exponential volume growth for which the heat kernel decays as $\exp(-cn^{\alpha})$ with $\alpha \in (0, 1)$ which can be taken arbitrarily close to 1. This, as mentioned above, cannot happen in the case of real Lie groups.

In a recent paper (cf. [10]) Pittet and Saloff-Coste established the lower bound $\varphi_{2n}(e) \geqslant c \exp(-Cn^{1/3})$, $n = 1, 2, \ldots$ for the large times asymptotic behaviour of the probabilities of return to the origin at even times $2n$, for random walks associated with finite symmetric generating sets of solvable groups of finite Prüfer rank. They asked in this paper (cf. [10, Section 8]) if a similar lower bound is available in the case of the semi-direct product $\mathrm{Sol}(\mathbb{Q}_p) = \mathbb{Q}_p^2 \rtimes \mathbb{Q}_p^*$ with $\mathbb{Q}_p^*$ dilating one of the two directions in $\mathbb{Q}_p^2$ and contracting the other one. Here, we shall give an answer to this problem.

We shall denote by $\mathbb{Q}_p$ the field of $p$-adic numbers (cf. [2,4]). We shall suppose that $\mathbb{Q}_p$ is equipped with its standard discrete valuation $|.|$. We let $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p^*, |x| = 1\}$, where $\mathbb{Q}_p^*$ is the multiplicative group of the field $\mathbb{Q}_p$. We denote by $\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x| \leqslant 1\}$. $\mathrm{d}x$ will denote the Haar measure on the additive group $\mathbb{Q}_p$ normalized by $\mathrm{d}x(\mathbb{Z}_p) = 1$ and $\mathrm{d}^*x$ will denote the Haar measure on $\mathbb{Q}_p^*$ normalized by $\mathrm{d}^*x(\mathbb{Z}_p^*) = 1$. Let us fix $k, l \geqslant 1$ and consider the semi-direct product

$$G = \mathbb{Q}_p^k \rtimes_\sigma (\mathbb{Q}_p^*)^l, \tag{1}$$

where the action $\sigma$ of $(\mathbb{Q}_p^*)^l$ on the vector space $\mathbb{Q}_p^k$ is defined by $k$ morphisms

$$\chi_1, \ldots, \chi_k : (\mathbb{Q}_p^*)^l \longrightarrow \mathbb{Q}_p^*. \tag{2}$$

More precisely, we assume that the multiplication in $G$ is given by

$$\begin{aligned} g.g' &= (x;y).(x';y') = (x + \sigma(y)x'; y.y') \\ &= (x_1 + \chi_1(y)x_1', x_2 + \chi_2(y)x_2', \ldots, x_k + \chi_k(y)x_k'; y_1.y_1', y_2.y_2', \ldots, y_l.y_l'), \end{aligned}$$

$$g = (x, y), \quad g' = (x', y') \in G; \quad x = (x_1, \ldots, x_k), \quad x' = (x_1', \ldots, x_k') \in \mathbb{Q}_p^k;$$

$$y = (y_1, \ldots, y_l), \quad y = (y_1', \ldots, y_l') \in (\mathbb{Q}_p^*)^l.$$

We shall denote by

$$\mathrm{d}^r g = \mathrm{d}x\, \mathrm{d}^*y = \mathrm{d}x_1 \ldots \mathrm{d}x_k\, \mathrm{d}^*y_1 \ldots \mathrm{d}^*y_l; \quad \mathrm{d}^l g = \mathrm{d}g = m(g)^{-1}\, \mathrm{d}^r g$$

the right and the left invariant Haar measure on $G$, where

$$m(g) = m(x, y) = |\chi_1(y)| \ldots |\chi_k(y)|, \quad g = (x, y) \in G$$

is the modular function normalized by $m(e) = 1$. The unimodularity of group $G$ is equivalent to the fact that

$$|\chi_1(y)| \ldots |\chi_k(y)| = 1, \quad y \in (\mathbb{Q}_p^*)^l. \tag{3}$$

We shall assume throughout that $G$ is unimodular and consider $\mathrm{d}\mu(g) = \varphi(g)\,\mathrm{d}g$ the symmetric, compactly supported, probability measure on $G$ defined by

$$\begin{aligned} \varphi(g) = {} & \alpha\phi_p(x_1) \ldots \phi_p(x_k)\phi_p(\chi_1(y)^{-1}x_1) \ldots \phi_p(\chi_k(y)^{-1}x_k) \\ & \times I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_1) \ldots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_l), \quad g = (x_1, \ldots, x_k; y_1, \ldots, y_l) \in G, \end{aligned} \tag{4}$$

where $\phi_p = I_{\mathbb{Z}_p}$ and where $\alpha > 0$ is an appropriate positive constant. The notation $I_A$ is used here to denote the characteristic function of a subset $A$. We shall call the random walk on $G$ induced by $\mu$ the simple random walk on $G$.

We shall assume throughout that $k \geqslant 2$ and that the $\chi_j$'s in (2) verify the condition

$$|\chi_j| \neq 1, \quad j = 1, \ldots, k. \tag{5}$$

This guarantees that the group $G$ defined by (1) is compactly generated (cf. [3]) and it is easy to see that $\mathrm{supp}(\mu)$ is a generating set of the group $G$, i.e.

$$G = \bigcup_{n \geqslant 1} (\mathrm{supp}(\mu))^n.$$

Assumption (5) implies that group $G$ is automatically of exponential growth (cf. [11]).

In this paper, we shall prove the following.

**Theorem 1.** *Let* $G = \mathbb{Q}_p^k \rtimes_\sigma (\mathbb{Q}_p^*)^l$ *and let* $\mu \in \mathbf{P}(G)$ *be as above. Let* $\mathrm{d}\mu^{*n}(g) = \mathrm{d}(\mu * \cdots * \mu)(g) = \varphi_n(g)\,\mathrm{d}g$ *denote the nth convolution power of* $\mu$. *Then*

$$\frac{1}{C} \exp(-c_1 n^{1/3}) \leqslant \varphi_{2n}(e) \leqslant C \exp(-c_2 n^{1/3}), \quad n = 1, 2, \ldots \tag{6}$$

*for some* $c_1$, $c_2$, $C > 0$ *independent of* $n$.

The main emphasis of the present work is on the lower bound. Although we provide a full proof of the upper bound in (6) the result itself is not new and follows from the general results of [7].

What is new is the probabilistic interpretation of the upper bound (in terms of maximal estimates related to the simple random walk on the integers). Along the way we obtain another bonus: an explicit formula for $\varphi_n(e)$. This formula is established in Section 2. The upper bound is proved in Section 3 and the lower bound in Section 4.

Throughout, we shall use the convention that the letters $C$ or $c$ indicate, possibly different, positive constants whose values are unimportant.

## 2. An explicit formula

The aim of this section is to obtain an explicit formula for $\varphi_n(e)$. We shall use the notation $x_i = (x_{i,1}, \ldots, x_{i,k})$ (resp. $y_i = (y_{i,1}, \ldots, y_{i,l})$) for $x_i \in \mathbb{Q}_p^k$, $i = 1, 2, \ldots$ (resp. $y_i \in (\mathbb{Q}_p^*)^l$, $i = 1, 2, \ldots$). Let $\mu$ be as in Theorem 1 and let $n = 1, 2, \ldots$ . We have

$$\begin{aligned}
\varphi_{n+1}(e) &= \langle \mu^{*n}, \varphi \rangle \\
&= \int_G \cdots \int_G \varphi(g_1 \ldots g_n)\,\mathrm{d}\mu(g_1) \ldots \mathrm{d}\mu(g_n) \\
&= \int_G \cdots \int_G \varphi(x_1 + \sigma(y_1)x_2 + \cdots + \sigma(y_1 \ldots y_{n-1})x_n, y_1 \ldots y_n) \\
&\quad \times \varphi(x_1, y_1) \ldots \varphi(x_n, y_n)\,\mathrm{d}x_1\,\mathrm{d}^*y_1 \ldots \mathrm{d}x_n\,\mathrm{d}^*y_n
\end{aligned}$$

$$= \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \left[ \int_{\mathbb{Q}_p^k} \cdots \int_{\mathbb{Q}_p^k} [\varphi(x_1 + x_2 + \cdots + x_n, y_1 \ldots y_n) \right.$$

$$\left. \times \varphi(x_1, y_1) \varphi(\sigma(y_1)^{-1} x_2, y_2) \ldots \varphi(\sigma(y_1 \ldots y_{n-1})^{-1} x_n, y_n)] \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \right]$$

$$\times \prod_{i=2}^{n} |\chi_1(y_1 \ldots y_{i-1})|^{-1} \ldots |\chi_k(y_1 \ldots y_{i-1})|^{-1} \, \mathrm{d}^* y_1 \ldots \mathrm{d}^* y_n,$$

where we used Fubini and the change of variable $\sigma(y_1 \ldots y_{i-1}) x_i \leftrightarrow x_i$, $i = 2, \ldots, n$. The unimodularity of $G$ then implies that

$$\varphi_{n+1}(e) = \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \left[ \int_{\mathbb{Q}_p^k} \cdots \int_{\mathbb{Q}_p^k} [\varphi(x_1 + x_2 + \cdots + x_n, y_1 \ldots y_n) \varphi(x_1, y_1) \right.$$

$$\times \varphi(\sigma(y_1)^{-1} x_2, y_2) \ldots \varphi(\sigma(y_1 \ldots y_{n-1})^{-1} x_n, y_n)]$$

$$\left. \times \mathrm{d}x_1 \ldots \mathrm{d}x_n \right] \mathrm{d}^* y_1 \ldots \mathrm{d}^* y_n$$

and if we use (4) we see that

$$\varphi_{n+1}(e) = \alpha^{n-1} \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \left[ \int_{\mathbb{Q}_p^k} \cdots \int_{\mathbb{Q}_p^k} [\varphi(x_1 + x_2 + \cdots + x_n, y_1 \ldots y_n) \right.$$

$$\times \varphi(x_1, y_1) \prod_{i=2}^{n} \phi_p(\chi_1(y_1 \ldots y_{i-1})^{-1} x_{i,1}) \ldots \phi_p(\chi_k(y_1 \ldots y_{i-1})^{-1} x_{i,k})$$

$$\left. \times \prod_{i=2}^{n} \phi_p(\chi_1(y_1 \ldots y_i)^{-1} x_{i,1}) \ldots \phi_p(\chi_k(y_1 \ldots y_i)^{-1} x_{i,k})] \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \right]$$

$$\times \prod_{i=2}^{n} I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,1}) \ldots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,l}) \, \mathrm{d}^* y_1 \ldots \mathrm{d}^* y_n$$

$$= \alpha^{n-1} \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \left[ \int_{\mathbb{Q}_p^k} \cdots \int_{\mathbb{Q}_p^k} \left[ \varphi(x_1 + x_2 + \cdots + x_n, y_1 \ldots y_n) \right. \right.$$

$$\times \varphi(x_1, y_1) \prod_{i=2}^{n} I_{|\chi_1(y_1 \ldots y_{i-1})|^{-1} \mathbb{Z}_p}(x_{i,1}) I_{|\chi_1(y_1 \ldots y_i)|^{-1} \mathbb{Z}_p}(x_{i,1})$$

$$\left. \left. \ldots I_{|\chi_k(y_1 \ldots y_{i-1})|^{-1} \mathbb{Z}_p}(x_{i,k}) I_{|\chi_k(y_1 \ldots y_i)|^{-1} \mathbb{Z}_p}(x_{i,k}) \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n \right]$$

$$\times \prod_{i=2}^{n} I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,1}) \ldots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,l}) \, \mathrm{d}^* y_1 \ldots \mathrm{d}^* y_n.$$

Hence (using again (4))

$$\varphi_{n+1}(e) = \alpha^{n+1} \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \left[ \int_{\mathbb{Q}_p^k} \cdots \int_{\mathbb{Q}_p^k} \left[ \prod_{j=1}^{k} I_{\max(1,|\chi_j(y_1 \dots y_n)|^{-1})\mathbb{Z}_p}(x_{1,j} + \cdots + x_{n,j}) \right.\right.$$
$$\times I_{\max(1,|\chi_1(y_1)|^{-1})\mathbb{Z}_p}(x_{1,1}) \dots I_{\max(1,|\chi_k(y_1)|^{-1})\mathbb{Z}_p}(x_{1,k})$$
$$\times \prod_{i=2}^{n} I_{|\chi_1(y_1 \dots y_{i-1})|^{-1} \max(1,|\chi_1(y_i)|^{-1})\mathbb{Z}_p}(x_{i,1})$$
$$\left.\left. \dots I_{|\chi_k(y_1 \dots y_{i-1})|^{-1} \max(1,|\chi_k(y_i)|^{-1})\mathbb{Z}_p}(x_{i,k}) \right] \mathrm{d}x_1 \dots \mathrm{d}x_n \right]$$
$$\times I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{1,1} \dots y_{n,1}) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{1,l} \dots y_{n,l})$$
$$\times \prod_{i=1}^{n} I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,1}) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,l})\, \mathrm{d}^*y_1 \dots \mathrm{d}^*y_n.$$

Let us observe now that for each $j = 1, \dots, k$

$$\int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} I_{\max(1,|\chi_j(y_1 \dots y_n)|^{-1})\mathbb{Z}_p}(x_{1,j} + \cdots + x_{n,j}) I_{\max(1,|\chi_j(y_1)|^{-1})\mathbb{Z}_p}(x_{1,j})$$
$$\times \prod_{i=2}^{n} I_{|\chi_j(y_1 \dots y_{i-1})|^{-1} \max(1,|\chi_j(y_i)|^{-1})\mathbb{Z}_p}(x_{i,j})\, \mathrm{d}x_{1,j}\, \mathrm{d}x_{2,j} \dots \mathrm{d}x_{n,j}$$
$$= \int_{\mathbb{Q}_p} (I_{\max(1,|\chi_j(y_1)|^{-1})\mathbb{Z}_p} * I_{|\chi_j(y_1)|^{-1} \max(1,|\chi_j(y_2)|^{-1})\mathbb{Z}_p} *$$
$$\cdots * I_{|\chi_j(y_1 \dots y_{n-1})|^{-1} \max(1,|\chi_j(y_n)|^{-1})\mathbb{Z}_p})(x) I_{\max(1,|\chi_j(y_1 \dots y_n)|^{-1})\mathbb{Z}_p}(x)\, \mathrm{d}x$$

where $*$ denotes the usual convolution in $\mathbb{Q}_p$ (cf. [12]). Therefore (cf. [12])

$$\int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} I_{\max(1,|\chi_j(y_1 \dots y_n)|^{-1})\mathbb{Z}_p}(x_{1,j} + \cdots + x_{n,j}) I_{\max(1,|\chi_j(y_1)|^{-1})\mathbb{Z}_p}(x_{1,j})$$
$$\times \prod_{i=2}^{n} I_{|\chi_j(y_1 \dots y_{i-1})|^{-1} \max(1,|\chi_j(y_i)|^{-1})\mathbb{Z}_p}(x_{i,j})\, \mathrm{d}x_{1,j}\, \mathrm{d}x_{2,j} \dots \mathrm{d}x_{n,j}$$
$$= \min(\max(1, |\chi_j(y_1)|^{-1}), \min_{2 \leqslant i \leqslant n} |\chi_j(y_1 \dots y_{i-1})|^{-1} \max(1, |\chi_j(y_i)|^{-1}))$$
$$\times \min(1, |\chi_j(y_1)|) \prod_{i=2}^{n} |\chi_j(y_1 \dots y_{i-1})| \min(1, |\chi_j(y_i)|)$$
$$\times \min[\max(\min(1, |\chi_j(y_1)|), \max_{2 \leqslant i \leqslant n} |\chi_j(y_1 \dots y_{i-1})| \min(1, |\chi_j(y_i)|);$$
$$\min(1, |\chi_j(y_1 \dots y_n)|))], \quad j = 1, \dots, k.$$

It then follows that

$$\varphi_{n+1}(e) = \alpha^{n+1} \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \prod_{j=1}^{k} \min[1; \min(\max(1, |\chi_j(y_1)|^{-1}),$$

$$\min_{2\leqslant i\leqslant n} |\chi_j(y_1 \dots y_{i-1})|^{-1} \max(1, |\chi_j(y_i)|^{-1}))$$

$$\times \min(1, |\chi_j(y_1 \dots y_n)|)] \prod_{i=1}^{n} \prod_{j=1}^{k} \min(1, |\chi_j(y_i)|)$$

$$\times I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{1,1} \dots y_{n,1}) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{1,l} \dots y_{n,l})$$

$$\times \prod_{i=1}^{n} I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,1}) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,l})\, \mathrm{d}^* y_1 \dots \mathrm{d}^* y_n,$$

where we used the unimodularity of $G$. Let us observe that

$$\int_{\mathbb{Q}_p^k} \varphi(x, y)\, \mathrm{d}x = \alpha \prod_{j=1}^{k} \min(1, |\chi_j(y)|) I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_1) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_l),$$

$$y = (y_1, \dots, y_l) \in (\mathbb{Q}_p^*)^l,$$

from which it follows that

$$\alpha \int_{(\mathbb{Q}_p^*)^l} \prod_{j=1}^{k} \min(1, |\chi_j(y)|) I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_1) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_l)\, \mathrm{d}^* y = 1.$$

Let us rewrite

$$\varphi_{n+1}(e) = \alpha \int_{(\mathbb{Q}_p^*)^l} \cdots \int_{(\mathbb{Q}_p^*)^l} \prod_{j=1}^{k} \min[1; \min(\max(1, |\chi_j(y_1)|^{-1}),$$

$$\min_{2\leqslant i\leqslant n} |\chi_j(y_1 \dots y_{i-1})|^{-1} \max(1, |\chi_j(y_i)|^{-1})) \min(1, |\chi_j(y_1 \dots y_n)|)]$$

$$\times I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{1,1} \dots y_{n,1}) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{1,l} \dots y_{n,l})$$

$$\times \alpha^n \prod_{i=1}^{n} I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,1}) \dots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_{i,l})$$

$$\times \prod_{i=1}^{n} \left( \prod_{j=1}^{k} \min(1, |\chi_j(y_i)|) \right) \mathrm{d}^* y_1 \dots \mathrm{d}^* y_n$$

and let us introduce a sequence $Y_1, Y_2, \ldots \in (\mathbb{Q}_p^*)^l$ of $(\mathbb{Q}_p^*)^l$-valued independent equidistributed random variables with distribution on $(\mathbb{Q}_p^*)^l$ given by

$$\mathbf{P}[Y_1 \in \mathrm{d}y] = \alpha\left(\prod_{j=1}^{k} \min(1, |\chi_j(y)|)\right) I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_1) \ldots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(y_l)\, \mathrm{d}^* y. \tag{7}$$

We finally obtain the formula

$$\begin{aligned} \varphi_{n+1}(e) = \alpha\mathbb{E}\Bigg(\prod_{j=1}^{k} & \min[1; \min(\max(1, |\chi_j(Y_1)|^{-1}), \\ & \min_{2\leqslant i\leqslant n} |\chi_j(Y_1 \ldots Y_{i-1})|^{-1} \max(1, |\chi_j(Y_i)|^{-1}))\min(1, |\chi_j(Y_1 \ldots Y_n)|)] \\ & \times I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(Y_{1,1} \ldots Y_{n,1}) \ldots I_{(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)}(Y_{1,l} \ldots Y_{n,l})\Bigg). \end{aligned} \tag{8}$$

## 3. Proof of the upper estimate

We now face the task of estimating the above expectation for the random variables $Y_1, Y_2, \ldots$. Let us denote by $\Pi_n = Y_1 \ldots Y_n$, $n = 1, 2, \ldots$, the random walk on $(\mathbb{Q}_p^*)^l$ starting at the origin and controlled by (7). The fact that the $Y_i$'s are supported in $(p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*) \times \cdots \times (p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)$ in (8) allows us to deduce that

$$\varphi_n(e) \leqslant C\mathbb{E}\left(\prod_{j=1}^{k} \min_{1\leqslant i\leqslant n} (\min(1, |\chi_j(\Pi_i)|^{-1}))\right), \quad n = 1, 2, \ldots .$$

We shall now use the fact that

$$(\mathbb{Q}_p^*)^l \cong \mathbb{Z}^l \times K,$$

where $K$ is identified to a compact subgroup of $(\mathbb{Q}_p^*)^l$ and fix $\tau_1, \ldots, \tau_l \in (\mathbb{Q}_p^*)^l$, so that each $y \in (\mathbb{Q}_p^*)^l$ can be uniquely represented in the form

$$y = \tau_1^{n_1} \ldots \tau_l^{n_l} k, \quad n_1, \ldots, n_l \in \mathbb{Z}, \quad k \in K. \tag{9}$$

For $i = 1, \ldots, k$, we have

$$|\chi_i(y)| = |\chi_i(\tau_1)|^{n_1} \ldots |\chi_i(\tau_l)|^{n_l} = p^{\gamma_{i,1}n_1 + \cdots + \gamma_{i,l}n_l},$$

where $y \in (\mathbb{Q}_p^*)^l$ is as in (9) and where the $l$-uple $(\gamma_{i,1}, \ldots, \gamma_{i,l}) \in \mathbb{Z}^l$ depends only on $\chi_i$. We shall denote by $L_1, L_2, \ldots, L_k$ the $k$ linear forms on $\mathbb{Z}^l$ defined by

$$\begin{aligned} & L_i(n) = L_i(n_1, \ldots, n_l) = \gamma_{i,1}n_1 + \cdots + \gamma_{i,l}n_l, \\ & n = (n_1, \ldots, n_l) \in \mathbb{Z}^l, \quad i = 1, \ldots, k. \end{aligned}$$

We shall use the projection

$$(\mathbb{Q}_p^*)^l \cong \mathbb{Z}^l \times K \longrightarrow \mathbb{Z}^l \tag{10}$$

and project the random walk $\Pi_j = Y_1 \ldots Y_j$, $j = 1, 2, \ldots$ on $\mathbb{Z}^l$. The random walk obtained via (10) is the simple (nearest-neighbour) symmetric random walk on $\mathbb{Z}^l$, $S_0, S_1, \ldots$ starting at $S_0 = 0$. With these notations we have

$$\begin{aligned}\varphi_n(e) &\leqslant C\mathbb{E}\left(\prod_{j=1}^{k} \min_{1\leqslant i\leqslant n}(\min(1, p^{-L_j(S_i)}))\right)\\ &\leqslant C\mathbb{E}\left(\prod_{j=1}^{k} \min_{1\leqslant i\leqslant n} p^{-L_j^+(S_i)}\right), \quad n = 1, 2, \ldots ,\end{aligned}$$

where $L_j^+(z) = \max(L_j(z), 0)$, $z \in \mathbb{Z}^l$, $j = 1, \ldots, k$ (we shall denote by $L^-(z) = \min(L_j(z), 0)$, $z \in \mathbb{Z}^l$). We then get

$$\begin{aligned}\varphi_n(e) &\leqslant C\mathbb{E}(p^{-\max_{1\leqslant i\leqslant n} L_1^+(S_i)} p^{-\max_{1\leqslant i\leqslant n} \sum_{j=2}^{k} L_j^+(S_i)})\\ &\leqslant C\mathbb{E}(p^{-\max_{1\leqslant i\leqslant n} L_1^+(S_i)} p^{-\max_{1\leqslant i\leqslant n} (\sum_{j=2}^{k} L_j(S_i))^+}).\end{aligned} \tag{11}$$

On the other hand, we have by the unimodularity of $G$ (cf. (3))

$$\left(\sum_{j=2}^{k} L_j\right)^+ = (-L_1)^+ = -L_1^-. \tag{12}$$

Putting together (11) and (12) we deduce that

$$\begin{aligned}\varphi_n(e) &\leqslant C\mathbb{E}(p^{-\max_{1\leqslant i\leqslant n} L_1^+(S_i)} p^{-\max_{1\leqslant i\leqslant n}(-L_1^-(S_i))})\\ &\leqslant C\mathbb{E}(p^{-\max_{1\leqslant i\leqslant n}(L_1^+(S_i) - L_1^-(S_i))}).\end{aligned}$$

This gives

$$\varphi_n(e) \leqslant C\mathbb{E}(p^{-\max_{1\leqslant i\leqslant n}|L_1(S_i)|}), \quad n = 1, 2, \ldots . \tag{13}$$

The upper estimate in (6) is an immediate consequence of (13) and the following estimate

$$\mathbf{P}[\max_{1\leqslant i\leqslant n} |L_1(S_i)| \leqslant \lambda] \leqslant C \exp\left(-c\frac{n}{\lambda^2}\right), \quad \lambda \geqslant 1, \quad n = 1, 2, \ldots . \tag{14}$$

To prove (14) it suffices to observe that $\xi_j = L_1(S_j - S_{j-1})$, $(j = 1, 2, \ldots)$ define a sequence of independent equidistributed and centered random variables on $\mathbb{Z}$ and repeat the argument given in [17, Lemma 2, p. 890].

## 4. Proof of the lower estimate

To prove the lower estimate in (6) we first observe as in Section 3, that in expectation (8), $Y_i \in (p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*) \times \cdots \times (p^{-1}\mathbb{Z}_p^* \cup p\mathbb{Z}_p^*)$, $i = 1, 2, \ldots$ . We have, therefore, the following lower bound:

$$\varphi_{n+1}(e) \geqslant c\mathbb{E}\left(\prod_{j=1}^{k} \min_{1\leqslant i\leqslant n}(\min(1, |\chi_j(\Pi_i)|^{-1}))I_{(p^{-1}\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}(\Pi_{n,1})\right.$$
$$\left.\dots I_{(p^{-1}\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}(\Pi_{n,l})\right), \quad n = 1, 2, \dots .$$

where the notations are as in Section 3. From this it follows that

$$\varphi_{n+1}(e) \geqslant c\mathbb{E}\left(\prod_{j=1}^{k} \min_{1\leqslant i\leqslant n}(\min(1, |\chi_j(\Pi_i)|^{-1}))I_{(p^{-1}\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}(\Pi_{n,1}) \dots I_{(p^{-1}\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}(\Pi_{n,l})\right.$$
$$\left.\times I[|\chi_j(Y_1 \dots Y_i)| \leqslant p^{n^{1/3}},\ j = 1, \dots, k,\ i = 1, \dots, n]\right)$$

and then

$$\varphi_{n+1}(e) \geqslant cp^{-Cn^{1/3}}\mathbb{E}(I[|\chi_j(Y_1 \dots Y_i)| \leqslant p^{n^{1/3}},\ j = 1, \dots, k,\ i = 1, \dots, n]$$
$$\times I_{(p^{-1}\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}(\Pi_{n,1}) \dots I_{(p^{-1}\mathbb{Z}_p^*\cup p\mathbb{Z}_p^*)}(\Pi_{n,l})), \quad n = 1, 2, \dots .$$

We project as in Section 3 the random walk $\Pi_j = Y_1 \dots Y_j$, $(j = 1, 2, \dots)$ on $\mathbb{Z}^l$ and we deduce that

$$\varphi_{2n}(e) \geqslant cp^{-Cn^{1/3}}\mathbf{P}(|S_j| \leqslant cn^{1/3},\ j = 0, \dots, 2n;\ S_{2n} = 0), \quad n = 1, 2, \dots , \tag{15}$$

where $S_j, j = 0, 1, \dots$ denote as in Section 3 the simple symmetric random walk on $\mathbb{Z}^l$ starting at $S_0 = 0$ and where $|.|$ denote the Euclidean norm on $\mathbb{Z}^l$. We claim that

$$\mathbf{P}(|S_j| \leqslant cn^{1/3},\ j = 0, \dots, n;\ S_{2n} = 0) \geqslant c\mathrm{e}^{-Cn^{1/3}}, \quad n = 1, 2, \dots \tag{16}$$

and the lower estimate in (6) is an immediate consequence of (15) and (16). Let us prove (16). This estimate is essentially an automatic consequence of the well-known estimate

$$\mathbf{P}[\max_{1\leqslant i\leqslant n} |S_i| \leqslant \lambda] \geqslant c \exp\left(-C\frac{n}{\lambda^2}\right), \quad \lambda \geqslant 1, \quad n = 1, 2, \dots . \tag{17}$$

Indeed, let $B_\lambda$ denote the ball of radius $\lambda \geqslant 1$ in $\mathbb{Z}^l$, that is, $B_\lambda = \{z \in \mathbb{Z}^l, |z| \leqslant \lambda\}$ and let $p_n^\lambda(x, y)$, $n = 1, 2, \dots$, $x, y \in B_\lambda$, denote the transition kernel corresponding to the simple random walk with killing outside of $B_\lambda$. We have

$$\mathbf{P}[|S_j| \leqslant \lambda,\ j = 0, \dots, n] = \sum_{y\in B_\lambda} p_n^\lambda(0, y), \quad n = 1, 2, \dots .$$

If we bare in mind that

$$p_{2n}^\lambda(0, 0) = \sum_{y\in B_\lambda} p_n^\lambda(0, y)p_n^\lambda(y, 0) = \sum_{y\in B_\lambda} (p_n^\lambda(0, y))^2, \quad n = 1, 2, \dots$$

we see that with an appropriate $C>0$ we have

$$C\lambda^{l/2}(p_{2n}^{\lambda}(0,0))^{1/2} \geqslant \sum_{y\in B_\lambda} p_n^{\lambda}(0,y), \quad n=1,2,\ldots$$

and, therefore,

$$C\lambda^{l}p_{2n}^{\lambda}(0,0) \geqslant (\mathbf{P}[|S_j|\leqslant\lambda,\ j=0,\ldots,n])^2, \quad \lambda\geqslant 1, \quad n=1,2,\ldots .$$

This combined with (17) gives

$$\lambda^{l}\mathbf{P}[|S_j|\leqslant\lambda,\ j=0,\ldots,2n;\ S_{2n}=0] \geqslant c\exp\left(-C\frac{n}{\lambda^2}\right), \tag{18}$$

$$\lambda\geqslant 1, \quad n=1,2,\ldots .$$

Choosing $\lambda=cn^{1/3}$ in (18) we deduce (16). This completes the proof of Theorem 1. □

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